

# On Special Semigroups Derived From an Arbitrary Semigroup<sup>1</sup>

Attila Nagy

Department of Algebra  
Budapest University of Technology and Economics  
P.O. Box 91  
1521 Budapest  
Hungary  
e-mail: nagyat@math.bme.hu

## Abstract

Let  $S$  be a semigroup,  $\Lambda$  a non-empty set and  $P$  a mapping of  $\Lambda$  into  $S$ . The set  $S \times \Lambda$  together with the operation  $\circ_P$  defined by  $(s, \lambda) \circ_P (t, \mu) = (sP(\lambda)t, \mu)$  form a semigroup which is denoted by  $(S, \Lambda, \circ_P)$ . Using this construction, we prove a common connection between the semigroups  $S$ ,  $S/\theta$  and  $S/\theta^* = (S/\theta)/(\theta^*/\theta)$ , where  $\theta$  and  $\theta^*/\theta$  are the kernels of the right regular representations of  $S$  and  $S/\theta$ , respectively. We also prove an embedding theorem for the semigroup  $(S, S/\theta, \circ_P)$ , where  $S$  is a left equalizer simple semigroup without idempotents, and  $P$  maps every  $\theta$ -class of  $S$  into itself.

## 1 Introduction

Let  $S$  be an arbitrary semigroup. It is known that the relation  $\theta$  on  $S$  defined by  $(a, b) \in \theta$  if and only if  $xa = xb$  for all  $x \in S$  is a congruence on  $S$ . This congruence is the kernel of the right regular representation  $\varphi : a \mapsto \varrho_a$  ( $a \in S$ ) of  $S$ ;  $\varrho_a : s \mapsto sa$  ( $s \in S$ ) is the inner right translations of  $S$  defined by  $a$ . For convenience, the semigroup  $\varphi(S) = S/\theta$  is also called the right regular representation of  $S$ . The  $\theta$ -class of  $S$  containing an element  $s \in S$  will be denoted by  $[s]_\theta$ .

Let  $\theta^*$  denote the congruence on the semigroup  $S$  for which  $\theta \subseteq \theta^*$  and  $\theta^*/\theta$  is the kernel of the right regular representation on  $S/\theta$ , where  $\theta^*/\theta$  is defined by  $([s]_\theta, [t]_\theta) \in \theta^*/\theta$  if and only if  $(s, t) \in \theta^*$  (see Theorem 5.6 of [5]). It is easy to see that  $(a, b) \in \theta^*$  if and only if  $(xa, xb) \in \theta$  for all  $x \in S$ , that is,  $sa = sb$  for all  $s \in S^2$  (see also [7] and [8]). The  $\theta^*$ -class of  $S$  containing an element  $s \in S$  will be denoted by  $[s]_{\theta^*}$ .

The right regular representation of semigroups plays an important role in the examination of semigroups. Here we cite some results of papers [1], [2] and [9], in which special types of semigroup are characterized by the help of their right regular representation.

A semigroup satisfying the identity  $ab = a$  (resp.  $ab = b$ ) is called a left zero (resp. right zero) semigroup. A semigroup is called a left group (resp. right

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group) if it is a direct product of a group and a left zero (resp. right zero) semigroup.

**Lemma 1** ([1]) *A semigroup  $S$  is a left group if and only if the right regular representation  $S/\theta$  of  $S$  is a group.*

A semigroup  $S$  is called an  $M$ -inversive semigroup ([10]) if, for each  $a \in S$ , there are elements  $x, y \in S$  such that  $ax$  and  $ya$  are middle units of  $S$ , that is,  $caxd = cd$  and  $cyad = cd$  is satisfied for all  $c, d \in S$ .

**Lemma 2** ([2]) *A semigroup  $S$  is  $M$ -inversive if and only if the right regular representation  $S/\theta$  of  $S$  is a right group.*

In [9], a semigroup  $S$  is called a left equalizer simple semigroup if, for arbitrary elements  $a, b \in S$ , the assumption  $x_0a = x_0b$  for some  $x_0 \in S$  implies  $xa = xb$  for all  $x \in S$ .

**Lemma 3** ([9]) *A semigroup  $S$  is left equalizer simple if and only if the right regular representation  $S/\theta$  of  $S$  is left cancellative.*

The previous lemmas show connections between  $S$  and  $S/\theta$ , in special cases. In this paper we would like to find a common connection between the semigroups  $S$ ,  $S/\theta$  and  $S/\theta^* = (S/\theta)/(\theta^*/\theta)$  in a general case. In our examination, the following construction plays an important role.

Let  $S$  be a semigroup,  $\Lambda$  an arbitrary set, and  $P$  is a mapping of  $\Lambda$  into  $S$ . It is easy to see that the set  $S \times \Lambda$  together with the operation  $(s, \lambda) \circ_P (t, \mu) = (sP(\lambda)t, \mu)$  is a semigroup. This semigroup is the dual of the semigroup constructed in Exercise 6 for §8.2 of [4]. This semigroup will be denoted by  $(S, \Lambda; \circ_P)$ .

In Section 2, we deal with the semigroups  $(S, \Lambda; \circ_P)$ . We show that the left cancellativity and the right simplicity of a semigroup  $S$  are inherited from  $S$  to the semigroup  $(S, \Lambda; \circ_P)$ .

In Section 3, the semigroups  $(S, S/\theta; \circ_P)$  are in the focus of our examination, where  $S$  is an arbitrary semigroup and  $P$  is an arbitrary mapping of the factor semigroup  $S/\theta$  into  $S$  with condition  $P([s]_\theta) \in [s]_\theta$ . We show that, for an arbitrary semigroup  $S$ , the right regular representation of the semigroup  $(S, S/\theta; \circ_P)$  is isomorphic to the semigroup  $(S/\theta^*, S/\theta; \circ_{P'})$ , where  $P'$  is the mapping of  $S/\theta$  into  $S/\theta^*$  defined by  $P'([s]_\theta) = [s]_{\theta^*}$ .

In Section 4, we prove an embedding theorem for the semigroups  $(S, S/\theta; \circ_P)$ , where  $S$  is an idempotent-free left equalizer simple semigroup. We prove that if  $S$  is a left equalizer simple semigroup without idempotent then the semigroup  $(S, S/\theta; \circ_P)$  can be embedded into a simple semigroup  $(S'', S/\theta; \circ_{P''})$  containing a minimal left ideal.

For notations and notions not defined here, we refer to [3], [4] and [6].

## 2 Hereditary properties

In this section we show that the left cancellativity and the right simplicity of a semigroup  $S$  are inherited from  $S$  to the semigroup  $(S, \Lambda; \circ_P)$ .

**Lemma 4** *If  $S$  is a left cancellative semigroup then the semigroup  $(S, \Lambda, \circ_P)$  is also left cancellative for any set  $\Lambda$  and any mapping  $P : \Lambda \mapsto S$ .*

**Proof.** Assume

$$(s, \lambda) \circ_P (t, \mu) = (s, \lambda) \circ_P (r, \tau)$$

for some  $s, t, r \in S$  and  $\lambda, \mu, \tau \in \Lambda$ . Then

$$(sP(\lambda)t, \mu) = (sP(\lambda)r, \tau)$$

from which it follows that

$$sP(\lambda)t = sP(\lambda)r \quad \text{and} \quad \mu = \tau$$

As  $S$  is left cancellative, we get  $t = r$ . Hence

$$(t, \mu) = (r, \tau).$$

Thus the semigroup  $(S, \Lambda, \circ_P)$  is left cancellative.  $\square$

**Lemma 5** *If  $S$  is a right simple semigroup then the semigroup  $(S; \Lambda; \circ_P)$  is also right simple for any set  $\Lambda$  and any mapping  $P : \Lambda \mapsto S$ .*

**Proof.** Let  $(s, \lambda), (t, \mu) \in (S, \Lambda, \circ_P)$  be arbitrary elements. As  $S$  is right simple,

$$sP(\lambda)S = S.$$

Then there is an element  $x \in S$  such that

$$sP(\lambda)x = t,$$

and so

$$(s, \lambda) \circ_P (x, \mu) = (sP(\lambda)x, \mu) = (t, \mu).$$

From this it follows that the semigroup  $(S, \Lambda, \circ_P)$  is right simple.  $\square$

**Corollary 1** *If  $S$  is a right group then the semigroup  $(S; \Lambda; \circ_P)$  is also a right group for any semigroup  $S$  and any mapping  $P : \Lambda \mapsto S$ .*

**Proof.** As a semigroup is a right group if and only if it is right simple and left cancellative, our assertion follows from Lemma 4 and Lemma 5.  $\square$

### 3 The right regular representation

In this section we deal with the right regular representation of semigroups  $(S, \Lambda, \circ_P)$  in that case when  $S$  is an arbitrary semigroup,  $\Lambda$  is the factor semigroup  $S/\theta$  and  $P$  is an arbitrary mapping of  $S/\theta$  into  $S$  with condition that  $P([s]_\theta) \in [s]_\theta$  for every  $s \in S$ . We note that the product  $\circ_P$  in the semigroup  $(S, S/\theta, \circ_P)$  does not depend on choosing  $P$ , because  $(s, [a]_\theta) \circ_P (t, [b]_\theta) = (sP([a]_\theta)t, [b]_\theta)$  for every  $s, t, a, b \in S$ , and  $sa' = sa''$  for every  $a', a'' \in [a]_\theta$ .

**Theorem 1** *Let  $S$  be an arbitrary semigroup. Let  $P$  be a mapping of  $S/\theta$  into  $S$  with condition  $P([a]_\theta) \in [a]_\theta$  for every  $[a]_\theta \in S/\theta$ . Let  $P'$  denote the mapping of  $S/\theta$  onto  $S/\theta^*$  defined by  $P'([a]_\theta) = [a]_{\theta^*}$ . Then the right regular representation of the semigroup  $(S, S/\theta; \circ_P)$  is isomorphic to the semigroup  $(S/\theta^*, S/\theta; \circ_{P'})$ .*

**Proof.** Let  $\theta^\bullet$  denote the kernel of the right regular representation of the semigroup  $(S, S/\theta, \circ_P)$ . Let  $\phi$  be the mapping of the factor semigroup  $(S, S/\theta, \circ_P)/\theta^\bullet$  onto the semigroup  $(S/\theta^*, S/\theta, \circ_{P'})$  defined by

$$\phi([(a, [b]_\theta)]_{\theta^\bullet}) = ([a]_{\theta^*}, [b]_\theta),$$

where  $[(a, [b]_\theta)]_{\theta^\bullet}$  denotes the  $\theta^\bullet$ -class of  $(S, S/\theta, \circ_P)$  containing the element  $(a, [b]_\theta)$  of  $(S, S/\theta, \circ_P)$ . To show that  $\phi$  is injective, assume

$$\phi([(a, [b]_\theta)]_{\theta^\bullet}) = \phi([(c, [d]_\theta)]_{\theta^\bullet})$$

for some  $[(a, [b]_\theta)]_{\theta^\bullet}, [(c, [d]_\theta)]_{\theta^\bullet} \in (S, S/\theta, \circ_P)/\theta^\bullet$ . Then

$$([a]_{\theta^*}, [b]_\theta) = ([c]_{\theta^*}, [d]_\theta)$$

and so

$$[a]_{\theta^*} = [c]_{\theta^*} \quad \text{and} \quad [b]_\theta = [d]_\theta.$$

Let  $x, \xi \in S$  be arbitrary elements. Then  $x\xi a = x\xi c$  and so

$$(x, [\xi]_\theta)(a, [b]_\theta) = (x\xi a, [b]_\theta) = (x\xi c, [d]_\theta) = (x, [\xi]_\theta)(c, [d]_\theta).$$

Hence

$$((a, [b]_\theta), (c, [d]_\theta)) \in \theta^\bullet,$$

that is,

$$[(a, [b]_\theta)]_{\theta^\bullet} = [(c, [d]_\theta)]_{\theta^\bullet}.$$

Thus  $\phi$  is injective. Consequently  $\phi$  is bijective. It remains to show that  $\phi$  is a homomorphism. Let

$$[(a, [b]_\theta)]_{\theta^\bullet}, [(c, [d]_\theta)]_{\theta^\bullet} \in (S, S/\theta, \circ_P)/\theta^\bullet$$

be arbitrary. Then

$$\begin{aligned}
\phi([(a, [b]_\theta)]_{\theta^\bullet} [(c, [d]_\theta)]_{\theta^\bullet}) &= \phi([(abc, [d]_\theta)]_{\theta^\bullet}) = \\
&= ([abc]_{\theta^*}, [d]_\theta) = ([a]_{\theta^*} [b]_{\theta^*} [c]_{\theta^*}, [d]_\theta) = \\
&= ([a]_{\theta^*}, [b]_\theta) \circ_{P'} ([c]_{\theta^*}, [d]_\theta) = \phi([(a, [b]_\theta)]_{\theta^\bullet}) \circ_{P'} \phi([(c, [d]_\theta)]_{\theta^\bullet}).
\end{aligned}$$

Hence  $\phi$  is a homomorphism, and so it is an isomorphism of the right regular representation of the semigroup  $(S, S/\theta, \circ_P)$  onto the semigroup  $(S/\theta^*, S/\theta, \circ_{P'})$ .  $\square$

**Corollary 2** *If  $S$  is a left group then the semigroup  $(S, S/\theta, \circ_P)$  is M-inversive.*

**Proof.** If  $S$  is a left group then  $S/\theta$  is a group by Lemma 1. As  $S^2 = S$ , we have  $\theta = \theta^*$ . Thus  $S/\theta^*$  is a group. By Corollary 1, the semigroup  $(S/\theta^*, S/\theta; \circ_{P'})$  is a right group and so, by Theorem 1 and Lemma 2,  $(S, S/\theta, \circ_P)$  is an M-inversive semigroup.  $\square$

**Corollary 3** *If  $S$  is an M-inversive semigroup then the semigroup  $(S, S/\theta, \circ_P)$  is M-inversive.*

**Proof.** If  $S$  is M-inversive then  $S/\theta$  is a right group by Lemma 2. As the kernel of the right regular representation of a right group is the identity relation,  $\theta^*/\theta$  is the identity relation on  $S/\theta$ . Then  $\theta^* = \theta$  and so  $S/\theta^*$  is a right group. By Corollary 1,  $(S/\theta^* \times S/\theta; \circ_{P'})$  is a right group. By Theorem 1 and Lemma 2,  $(S, S/\theta, \circ_P)$  is an M-inversive semigroup.  $\square$

**Corollary 4** *If  $S$  is a left equalizer simple semigroup then the semigroup  $(S, S/\theta, \circ_P)$  is left equalizer simple.*

**Proof.** Let  $S$  be a left equalizer simple semigroup. Then, by Lemma 3,  $S/\theta$  is a left cancellative semigroup. As the right regular representation of a left cancellative semigroup is the identity relation,  $\theta^*/\theta$  is the identity relation on  $S/\theta$ . From this it follows that  $\theta^* = \theta$  and so  $S/\theta^*$  is a left cancellative semigroup. By Lemma 4,  $(S/\theta^*, S/\theta, \circ_{P'})$  is a left cancellative semigroup. By Theorem 1 and by Lemma 3, the semigroup  $(S, S/\theta, \circ_P)$  is M-inversive.  $\square$

## 4 An embedding theorem

In this section we deal with such semigroups  $(S, S/\theta, \circ_P)$  in which  $S$  is an idempotent-free left equalizer simple semigroup.

**Theorem 2** *If  $S$  is a left equalizer simple semigroup without idempotents then the semigroup  $(S, S/\theta, \circ_P)$  can be embedded into a simple semigroup  $(S'', S/\theta, \circ_{P''})$  containing a minimal left ideal.*

**Proof.** Let  $S$  be a left equalizer simple semigroup without idempotent. Then, by Theorem 8.19 of [4], there is an embedding  $\tau$  of  $S$  into a left simple semigroup  $S''$  without idempotents. Consider the semigroup  $(S'', S/\theta, \circ_{P''})$ , where the mapping  $P'' : S/\theta \mapsto S''$  is defined by  $P''([s]_\theta) = \tau(P[s]_\theta)$ . By the dual of Exercise 6 for §8.2 of [4],  $(S'', S/\theta, \circ_{P''})$  is a simple semigroup containing a minimal left ideal. We show that

$$\Phi : (a, [b]_\theta) \mapsto (\tau(a), [b]_\theta)$$

is an embedding of the semigroup  $(S, S/\theta, \circ_P)$  into the semigroup  $(S'', S/\theta, \circ_{P''})$ .

First we show that  $\Phi$  is injective. Assume

$$\Phi((a, [b]_\theta)) = \Phi((c, [d]_\theta))$$

for some  $a, b, c, d \in S$ . Then

$$\tau(a) = \tau(c) \quad \text{and} \quad [b]_\theta = [d]_\theta.$$

As  $\tau$  is injective, we get

$$(a, [b]_\theta) = (c, [d]_\theta).$$

Next we show that  $\Phi$  is a homomorphism. Let

$$(a, [b]_\theta), (c, [d]_\theta) \in (S, S/\theta, \circ_P)$$

be arbitrary elements. Then

$$\begin{aligned} \Phi((a, [b]_\theta) \circ_P (c, [d]_\theta)) &= \Phi((aP([b]_\theta)c, [d]_\theta)) = (\tau(aP([b]_\theta)c), [d]_\theta) = \\ &= (\tau(a)\tau(P([b]_\theta))\tau(c), [d]_\theta) = (\tau(a)P''([b]_\theta)\tau(c), [d]_\theta) = \\ &= (\tau(a), [b]_\theta) \circ_{P''} (\tau(c), [d]_\theta) = \Phi((a, [b]_\theta)) \circ_{P''} \Phi((c, [d]_\theta)) \end{aligned}$$

and so  $\Phi$  is a homomorphism. Consequently  $\Phi$  is an embedding of the semigroup  $(S, S/\theta, \circ_P)$  into the simple semigroup  $(S'', S/\theta, \circ_{P''})$  containing a minimal left ideal.  $\square$

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